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The double-space parking problem

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Abstract

A double-space parking problem is studied for a parking lot of size M accommodating both private cars and buses. Upon arrival, a private car is either admitted to the parking lot, occupying a single spot, or waits in line until a spot becomes available. An arriving bus occupies double spots and is admitted only if there are at least two free spots. It balks from the system otherwise. The inflow is governed by two independent Poisson streams, with rates λ_C for cars and λ_B for buses. The sojourn time of a car or a bus inside the parking lot is exponentially distributed with parameters μ_C and μ_B, respectively. The problem is formulated as a QBD process and analyzed via matrix geometric methods. Various performance measures are calculated, including mean number of cars inside, and outside, the parking lot; mean number of buses in the system; and the probability that an arriving bus is blocked. The dichotomy whether to split the M-spot lot into two separate lots, one for cars, the other for buses, is studied and the optimal split is calculated. Numerical results are presented via graphs. Finally, it is shown that from the point of view of the parking lot owner, it is equivalent to either charge a fixed entrance fee or charge per-time unit of usage.

Keywords Queueing · Random number of servers · QBD process · Matrix geometric · Profit maximization

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1 Introduction

This paper investigates the queueing problem of a parking lot serving two types of cars: private cars and buses. A car occupies a single parking spot, while a bus requires double parking spots. This type of problem falls into the area of multi-server queueing systems in which different customers require a random number of servers serving simultaneously. Such problems have been addressed extensively in the literature [see, e.g., (Brill and Green 1984; Fletcher et al. 1986; Federgruen and Green 1984; Green 1980, 1981; Kaufman 1981; Rumyantsev and Morozov 2017)]. The studied models have been applied in various areas such as shared resource environment [see, e.g., (Kaufman 1981)], satellite communication systems operating under a frequency division multiple access scheme (FDMA), and under time division multiple access scheme (TDMA) for satellites (Fletcher et al. 1986). An M/M/m-type model where servers are assigned to the same customer but do not end service simultaneously is discussed in Green (1980), where analytic expressions for the distribution of a customer's waiting time in the queue, as well as the distribution of the number of busy servers, are obtained. The same model is analyzed in Federgruen and Green (1984), assuming that each sever has a general service time distribution. The queue-length distribution is approximated. A closed queueing system with a fixed number of customers, where customers who finish service are fed back to the end of the queue, is studied in Fletcher et al. (1986), and the distribution of the number of busy servers, as well as queue-length distribution, are analyzed. Stability condition for a multi-server model with simultaneous service was derived in Rumyantsev and Morozov (2017).

The current paper concentrates on the analysis of the so-called double-space parking problem. Specifically, we consider a parking lot having M parking spots that accommodates both cars and buses. The problem is formulated (Sect. 2) as a quasi-birth-and-death (QBD) process with a two-dimensional state space. One dimension counts the number of buses inside the parking lot; the other counts the total number of cars in the system (both inside and outside the lot). In Sect. 3, matrix geometric analysis is applied [see (Neuts 1981; Hanukov et al. 2018, 2017, 2018; Hanukov and Yechiali 2020)], and the system's stability condition is determined, along with the system's 2-dimensional steady-state probabilities. Moreover, the entries of the so-called rate matrix R are directly calculated without applying the commonly used successive substitution procedure. Performance measures, such as the mean number of cars and mean number of buses, are calculated in Sect. 4. Numerical results are presented via graph in Sect. 5. The parking's lot optimal size is calculated in Sect. 6, and equivalence of two charging methods used in parking lots is discussed. The question of whether or not to split the parking lot into two separate lots, one for cars, the other for buses, is investigated in Sect. 7. Conclusions are discussed in Sect. 8.

2 The model

We consider a parking lot (PL) that accommodating two types of cars: private and buses. The number of available spaces is M (assumed even for simplicity), and the arrival processes of private cars and of buses are two independent Poisson processes with rates λ_C and λ_B, respectively. A private car occupies a single parking spot, while a bus requires double parking spots. A bus enters the PL if and only if there are at least two empty spots, or leaves otherwise. This assumption is made partly by the practice of bus companies that prefer buses to move to other locations and complete service there, rather than letting buses park and wait for a future task in the area where they have just completed service. This assumption is also made for analytical reasons.

Private cars may queue and wait at the entrance of the PL if all M spaces are occupied. The net parking time of a car (bus) is a random variable, exponentially distributed with parameter μ_C (μ_B, respectively), independent of each other. We assume that if a bus arrives and there are two or more empty spots, at least two of them are next to each other. This assumption is supported by the practice that cars may be moved from one spot to the other by the PL operator (this is a common practice in PL management where cars are moved from one place to another either by human operators or automatically). Furthermore, in order to utilize efficiently the parking lot's multi-use by both private cars and buses, the parking lot's area can be partitioned by signs into two separate areas, such that buses will mostly park in one part, each bus occupying two spots, while private cars will mostly occupy the other part. In cases that one part is fully used, cars or buses are allowed to park in the other part. This arrangement will minimize the number of car movements within the lot, practically justifying our model assumption.

Another example for double-space parking is a container shipping company that operates a yard with a limited storage area. Standard shipping containers are 8ft (2.44 m) wide, 8.5 ft (2.59 m) high and come in two lengths; 20ft (6.1 m) and 40ft (12.2 m). The former is commonly called twenty equivalent unit (TEU) container, while the latter is called forty equivalent unit (FEU) container. To store a TEU container, only a single-unit area is needed, whereas to store an FEU container, two unit areas are required. It is relatively easy to move containers around, and therefore, whenever needed, TEU containers can be readily moved around, making room for a FEU container.

Formulation Consider the system in steady state. Let L_B denote the number of buses actually parking in the lot, and let L_C denote the total number of cars (parking and queueing) in the system. The system's state is defined as a two-dimensional random vector (L_B, L_C). Let $\pi_{i,j} = P(L_B = i, L_C = j), \quad (0 \leq i \leq \frac{M}{2}, \quad 0 \leq j)$ denote the corresponding steady-state probabilities. Clearly, if $2i + j > M$, then the number of cars queueing outside the PL is $2i + j - M$. The system can be formulated and analyzed as a continuous-time quasi-birth-and-death (QBD) process, as described in the sequel. The transition rate diagram of the system's states is depicted in Fig. 1.

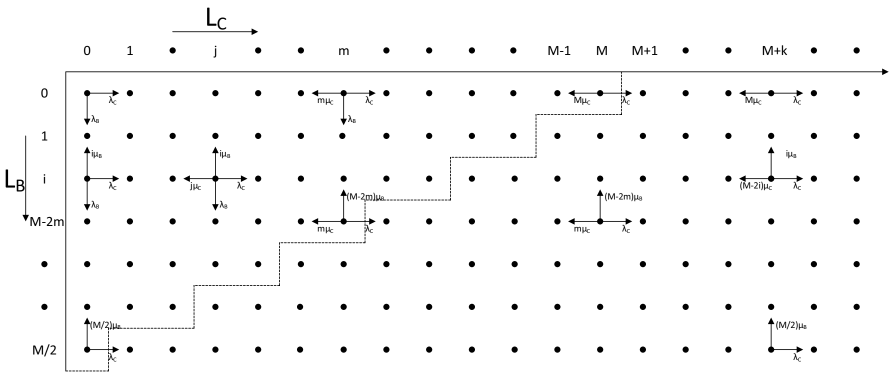

Fig. 1 Transition-rate diagram for (L_B, L_C): The states to the right of the dashed line indicate the states where cars are queueing at the entrance to the parking lot

3 Analysis: matrix geometric

The analysis of 2-dimensional quasi-birth-and-death (QBD) processes (such as the one depicted in Fig. 1) is thoroughly detailed in Neuts (1981). It enables matrix-type computation of the system's state probabilities as will be described in the sequel. Another method to analyze such processes is by defining and using $M/2 + 1$ probability generating function, each for every line in Fig. 1. This method requires the calculation of the roots between 0 and 1 of a given finite-dimensional matrix that its entries are composed from the system's parameter. In our problem, the use of matrix-geometric methods seems to be more efficient. For more details see, e.g., Hanukov and Yechiali (2020). Consequently, consider again the state space $\{(i,j)\}$ denoting i buses and j cars in the system, $0 \le i \le \frac{M}{2}$, $\quad 0 \le j$. When $L_B = i$ and $L_C = j$, we say that the system is in phase i and level j. By arranging the states in a lexicographic order, $(0,0),(1,0),\dots,(\frac{M}{2},0);(0,1),(1,1),\dots,(\frac{M}{2},1);,\dots,(0,m),(1,m),\dots,(\frac{M}{2},m);,\dots,(0,M+k),$ $(1,M+k),\dots,(\frac{M}{2},M+k);\dots k=0,1,2,\dots$, the 'generator' Q of the QBD process is given by

$$Q = \begin{bmatrix} B_{0,0} & A_0 & \dots & 0 & 0 & 0 & \dots & 0 \\ B_{1,0} & B_{1,1} & A_0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & B_{M-1,M-2} & B_{M-1,M-1} & A_0 & 0 & \dots & 0 \\ 0 & \dots & 0 & A_2 & A_1 & A_0 & 0 & \dots \\ 0 & \dots & 0 & 0 & A_2 & A_1 & A_0 & 0 \\ \vdots & \dots & 0 & 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & \ddots & \ddots & & & \end{bmatrix}$$

where the $(\frac{M}{2}+1) \times (\frac{M}{2}+1)$ square matrices $B_{j,j}, B_{j,j-1}$, A_0, A_1 and A_2 are given below:

$$B_{j,j} = \begin{bmatrix} \alpha_{0,j} & \lambda_B & 0 & \dots & 0 \\ \mu_B & \alpha_{1,j} & \ddots & \dots & 0 \\ 0 & 2\mu_B & \ddots & I_{\{2i+j<M-1\}}\lambda_B & \vdots \\ \vdots & 0 & \ddots & \alpha_{M/2-1,j} & 0 \\ 0 & \dots & 0 & \frac{M}{2}\mu_B & -(\lambda_C + \frac{M}{2}\mu_B) \end{bmatrix}$$

where $\alpha_{i,j} = -(\min\{(M-2i), j\}\mu_C + \lambda_C + I_{\{2i+j<M-1\}}\lambda_B + i\mu_B), \quad 0 \le i \le \frac{M}{2} - 1, 1 \le j \le M-2$.

In particular,

$$B_{0,0} = \begin{bmatrix} \alpha_{0,0} & \lambda_B & 0 & \dots & 0 \\ \mu_B & \alpha_{1,0} & \lambda_B & \dots & 0 \\ 0 & 2\mu_B & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \alpha_{M/2-1,0} & \lambda_B \\ 0 & \dots & 0 & \frac{M}{2}\mu_B & -(\lambda_C + \frac{M}{2}\mu_B) \end{bmatrix}$$

where $\alpha_{i,0} = -(\lambda_C + \lambda_B + i\mu_B), \quad 0 \le i \le \frac{M}{2} - 1.$

$$B_{M-1,M-1} = \begin{bmatrix} \gamma_0 & 0 & 0 & \dots & 0 \\ \mu_B & \gamma_1 & 0 & \dots & 0 \\ 0 & 2\mu_B & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \gamma_{M/2-1} & 0 \\ 0 & \dots & 0 & \frac{M}{2}\mu_B & \gamma_{M/2,j} \end{bmatrix}$$

where $\gamma_i = -(\min\{(M-2i), M-1\}\mu_C + \lambda_C + i\mu_B), \quad 0 \le i \le \frac{M}{2}.$

$$B_{j,j-1} = \begin{bmatrix} \min\{M,j\}\mu_C & 0 & 0 & \dots & 0 \\ 0 & \min\{M-2,j\}\mu_C & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \min\{2,j\}\mu_C & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

where $0 \le i \le \frac{M}{2}, 0 \le j \le M$.

$$A_0 = \begin{bmatrix} \lambda_C & 0 & 0 & 0 & \dots \\ 0 & \lambda_C & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_C & 0 \\ 0 & 0 & \dots & 0 & \lambda_C \end{bmatrix}$$

$$A_2 = \begin{bmatrix} M\mu_C & 0 & 0 & 0 & \dots \\ 0 & (M-2)\mu_C & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 2\mu_C & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} -(M\mu_C + \lambda_C) & 0 & 0 & 0 & \dots \\ \mu_B & -((M-2)\mu_C + \lambda_C + \mu_B) & 0 & 0 & \dots \\ 0 & 2\mu_B & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & -(2\mu_C + \lambda_C + (\frac{M}{2}-1)\mu_B) & 0 \\ 0 & \dots & 0 & \frac{M}{2}\mu_B & -(\lambda_C + \frac{M}{2}\mu_B) \end{bmatrix}$$

By Neuts (1981), the generator Q satisfies

$$\bar{p}Q = \bar{0}, \quad \bar{p}\bar{e} = 1$$

where $\bar{0}$ is a row vector with all its elements equal to zero, $\bar{e}$ is a column vector with all its entries equal to 1, and $\bar{p} = (\bar{p}_0, \bar{p}_1, \bar{p}_2, \dots \bar{p}_j, \dots)$, where

$$\bar{p}_j = (p_{0j}, p_{1j}, p_{2j}, \dots, p_{\frac{M}{2}j}), \quad j \geq 0.$$

The $(\frac{M}{2}+1)$-dimensional probability vector $\bar{p}_j$ denotes the phase probabilities when the system is in level j, presenting the states of column j in Fig. 1.

Consider now the matrix $A = A_0 + A_1 + A_2$, given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \mu_B & -\mu_B & 0 & 0 & \dots \\ 0 & 2\mu_B & -2\mu_B & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & (\frac{M}{2}-1)\mu_B & -(\frac{M}{2}-1)\mu_B) & 0 \\ 0 & \dots & 0 & \frac{M}{2}\mu_B & -\frac{M}{2}\mu_B \end{bmatrix}$$

The matrix A defines a one-dimensional absorbing birth-and-death Markovian process that its transition-rate diagram is depicted in Fig. 2. This underlying process represents a zero-buffer M/M/$\frac{M}{2}$ Markovian queue with arrival rate λ_B and $M/2$ parallel servers, each serving at rate μ_B.

Let $\bar{\pi} = (\pi_0, \pi_1, \pi_2, \dots, \pi_{\frac{M}{2}})$ denote the invariant probability vector of A. Then $\bar{\pi}$ satisfies $\bar{\pi}A = \bar{0}$, $\bar{\pi}\bar{e} = 1$. This leads to $\bar{\pi} = (1, 0, \dots, 0)$.

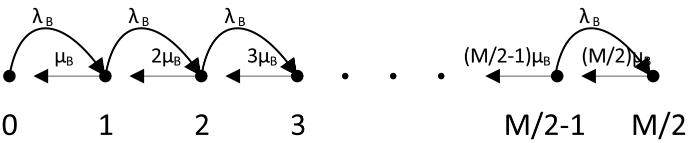

Fig. 2 Transition-rate diagram for the matrix A

The stability condition of the system (Neuts 1981) is $\bar{\pi}A_2\bar{e} > \bar{\pi}A_0\bar{e}$, which translates into

$$M\mu_C > \lambda_C. \tag{1}$$

This follows since the bus processes regulate itself by the 'blocked and leave' policy, while the car's buffer is unlimited.

Remark 3.1 The stability condition follows directly from a recent result by Hanukov and Yechiali (2020), stating that if A_0, A_1 and A_2 are each lower triangular, then the stability condition is given directly by $a_2^{0,0} > a_0^{0,0}$, where $A_n = [a_n^{ij}]$, $n = 0, 1, 2$.

From (Neuts 1981), the level probability vectors are given by

$$\bar{p}_{M+j} = \bar{p}_M R^j \quad j \geq 0,$$

where R is the minimal non-negative solution of the matrix quadratic equation

$$A_0 + RA_1 + R^2 A_2 = 0. \tag{2}$$

Theorem 3.2 *The matrix $R = [r_{ij}]$ is given by*

$$r_{ij} = \begin{cases} \frac{[(M-2i)\mu_C+\lambda_C+i\mu_B]-\sqrt{[(M-2i)\mu_C+\lambda_C+i\mu_B]^2-4\mu_C(M-2i)\lambda_C}}{2\mu_C(M-2i)} & j = i < M/2, \\ \frac{\lambda_C}{\lambda_C+(M/2)\mu_B} & j = i = M/2 \\ \frac{r_{i,j+1}\cdot(j+1)\mu_B+\sum_{k=j+1}^{i-1} r_{ik}r_{kj}\mu_C(M-2j)}{(M-2j)\mu_C(1-r_{ii}-r_{jj})+\lambda_C+j\mu_B} & j < i, \\ 0 & i < j, \end{cases} \tag{3}$$

where $0 \leq i, j \leq \frac{M}{2}$.

Proof Since A_0, A_1 and A_2 are lower triangular, it follows [see (Hanukov and Yechiali 2020)] that the matrix R is also lower triangular. Hence, the elements of the matrix

$$\widetilde{C} = [\widetilde{c}_{ij}] = A_0 + RA_1 + R^2 A_2$$

are given by

$$\tilde{c}_{ij} = \begin{cases} \lambda_C - r_{ii}[(M-2i)\mu_C + \lambda_C + i\mu_B] + \sum\limits_{k=0}^{M/2} r_{ik}r_{ki}\mu_C(M-2i) & j = i, \\ -r_{ij}[(M-2j)\mu_C + \lambda_C + j\mu_B] + r_{i,j+1}\cdot(j+1)\mu_B + \sum\limits_{k=0}^{M/2} r_{ik}r_{kj}\mu_C(M-2j) & j < i, \\ 0 & i < j, \end{cases}$$

$$= \begin{cases} \lambda_C - r_{ii}[(M-2i)\mu_C + \lambda_C + i\mu_B] + r_{ii}^2\mu_C(M-2i) & j = i, \\ -r_{ij}[(M-2j)\mu_C + \lambda_C + j\mu_B] + r_{i,j+1}\cdot(j+1)\mu_B + \sum\limits_{k=0}^{M/2} r_{ik}r_{kj}\mu_C(M-2j) & j < i, \\ 0 & i < j. \end{cases}$$

$$= \begin{cases} \lambda_C - r_{ii}[(M-2i)\mu_C + \lambda_C + i\mu_B] + r_{ii}^2\mu_C(M-2i) & j = i, \\ -r_{ij}[(M-2j)\mu_C + \lambda_C + j\mu_B] + r_{i,j+1}\cdot(j+1)\mu_B + \sum\limits_{k=j}^{i} r_{ik}r_{kj}\mu_C(M-2j) & j < i, \\ 0 & i < j. \end{cases}$$

Now, using (2) and setting $\tilde{c}_{ij} = 0$, we obtain Eq. (3), with no need to apply the commonly used successive substitutions approach to calculate R. Other cases where the matrix R is determined explicitly can be found in Hanukov et al. (2018, 2017, 2018). □

The probability vectors $\bar{p}_j$ for $j = 0, 1, 2, \dots, M$ are calculated by solving the following linear system of equations

$$\begin{aligned} &\bar{p}_0B_{00} + \bar{p}_1B_{10} = \bar{0}, \\ &\bar{p}_0A_0 + \bar{p}_1B_{11} + \bar{p}_2B_{21} = \bar{0}, \\ &\bar{p}_1A_0 + \bar{p}_2B_{22} + \bar{p}_3B_{32} = \bar{0}, \\ &\qquad\vdots \\ &\bar{p}_{M-2}A_0 + \bar{p}_{M-1}B_{M-1,M-1} + \bar{p}_MA_2 = \bar{0}, \\ &\bar{p}_{M-1}A_0 + \bar{p}_MA_1 + \bar{p}_MRA_2 = \bar{0}, \end{aligned}$$

together with the normalizing condition:

$$\left(\sum_{j=0}^{M-1} \bar{p}_j + \bar{p}_M[I-R]^{-1}\right)\bar{e} = 1.$$

4 Performance measures

In this section we calculate the mean number of cars and the mean number of buses in the system.

The mean number of cars is given by

$$\begin{aligned}
E[L_C] &= \sum_{j=0}^{M-1} j(\bar{p}_j\bar{e}) + \sum_{j=M}^{\infty} j(\bar{p}_j\bar{e}) \\
&= \sum_{j=1}^{M-1} j(\bar{p}_j\bar{e}) + \sum_{k=0}^{\infty} (M+k)(\bar{p}_{M+k}\bar{e}) \\
&= \sum_{j=1}^{M-1} j(\bar{p}_j\bar{e}) + M\sum_{k=0}^{\infty} (\bar{p}_{M+k}\bar{e}) + \sum_{k=0}^{\infty} k(\bar{p}_{M+k}\bar{e}) \\
&= \sum_{j=1}^{M-1} j(\bar{p}_j\bar{e}) + M\bar{p}_M(\sum_{k=0}^{\infty}(R^k)\bar{e} + \bar{p}_M(\sum_{k=0}^{\infty}(kR^k)\bar{e} \\
&= \sum_{j=1}^{M-1} j(\bar{p}_j\bar{e}) + M\bar{p}_M[I-R]^{-1}\bar{e} + \bar{p}_M\Big([I-R]^{-2} - [I-R]^{-1}\Big)\bar{e} \\
&= \sum_{j=1}^{M-1} j(\bar{p}_j\bar{e}) + (M-1)\bar{p}_M[I-R]^{-1}\bar{e} + \bar{p}_M[I-R]^{-2}\bar{e}.
\end{aligned}$$

The sojourn time (waiting and parking) of a car in the system is denoted by W_C. By Little's law

$$E[W_C] = \frac{E[L_C]}{\lambda_C}.$$

The mean queueing time of a car outside the PL is

$$E[W_C^q] = E[W_C] - \frac{1}{\mu_C}$$

and the mean queue size of outside waiting cars is

$$E[L_C^q] = \lambda_C[W_C^q] = E[L_C] - \frac{\lambda_C}{\mu_C}$$

The mean number of buses, $E[L_B]$, is calculated as follows. Let $\bar{z} = (0, 1, 2, 3, \ldots, \frac{M}{2} - 1, \frac{M}{2})^T$ be a column vector. Then, $E[L_B]$ is given by

$$\begin{aligned}
E(L_B) &= \sum_{j=0}^{\infty} \bar{p}_j\bar{z} = \sum_{j=0}^{M-1} \bar{p}_j\bar{z} + \sum_{k=0}^{\infty}(\bar{p}_M R^k)\bar{z} \\
&= \sum_{j=0}^{M-1} \bar{p}_j\bar{z} + \bar{p}_M[I-R]^{-1}\bar{z}.
\end{aligned}$$

Let L_{CI} denote the number of cars inside the PL. Then

$$E[L_{CI}] = \sum_{i=0}^{\frac{M}{2}} \sum_{j=0}^{M-2i} j p_{i,j} + \sum_{i=0}^{\frac{M}{2}} \sum_{j=M-2i+1}^{\infty} (M-2i) p_{i,j}. \tag{4}$$

On the other hand, also by Little's law,

$$E[L_{CI}] = \frac{\lambda_C}{\mu_C}.$$

This follows since in steady state all cars eventually enter the PL and the mean sojourn time of a car in the PL is $\frac{1}{\mu_C}$. Clearly, $E[L_C^q] = E[L_C] - E[L_{CI}]$.

Let OCU denote the total number of parking spots occupied by both cars and buses. Then,

$$E[OCU] = E[L_{CI}] + 2E[L_B].$$

Another interesting measure is L_{total}, the virtual total number of parking spots that would have been occupied if one counts each queueing car outside the parking lot as occupying a spot. Then,

$$E[L_{total}] = E[L_C] + 2E[L_B].$$

5 Numerical results

Figures 3, 4, 5 and 6 exhibit graphs showing the relative behavior of the above performance measures. Figure 3 depicts the values of $2E[L_B]$, $E[L_C]$, $E[L_{CI}]$ and $E[OCI]$ as functions of λ_C, for $M = 8, \mu_B = 1, \mu_C = 1, \lambda_B = 2$. It is seen that, as

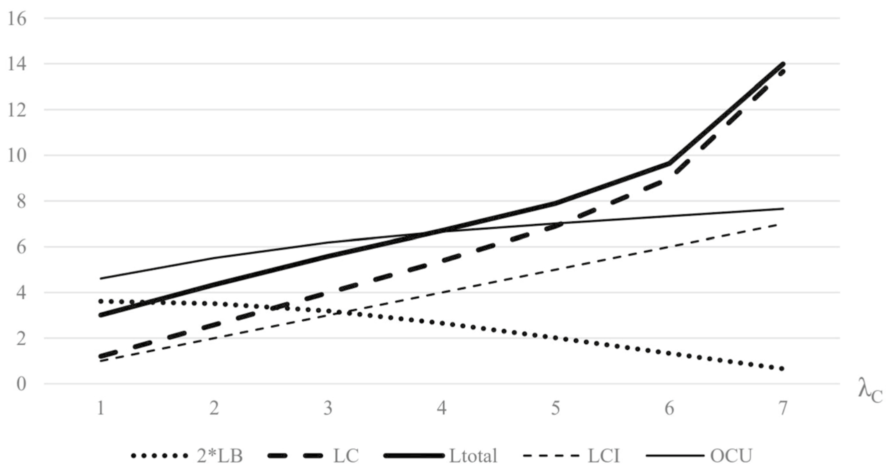

Fig. 3 Number of cars and buses as functions of λ_C when $M = 8, \mu_B = 1, \mu_C = 1, \lambda_B = 2$

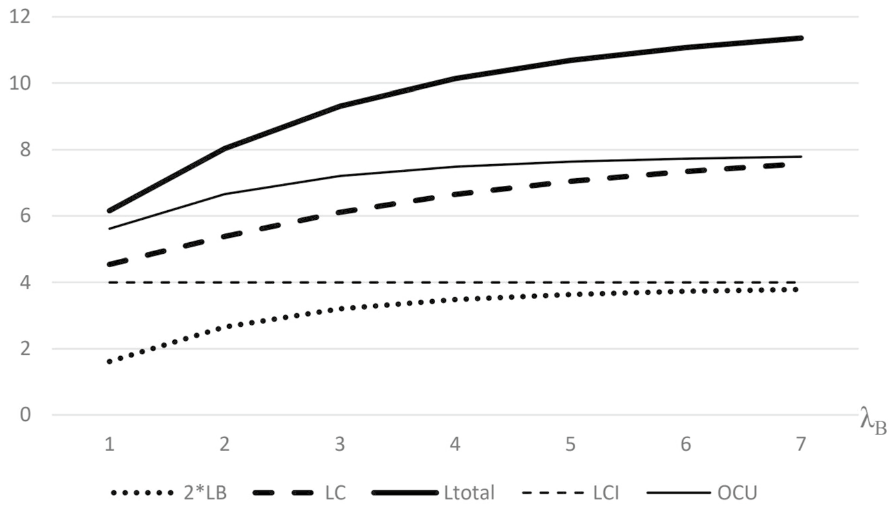

Fig. 4 Number of cars and buses as functions of λ_B when $M = 8, \mu_B = 1, \mu_C = 1, \lambda_C = 4$

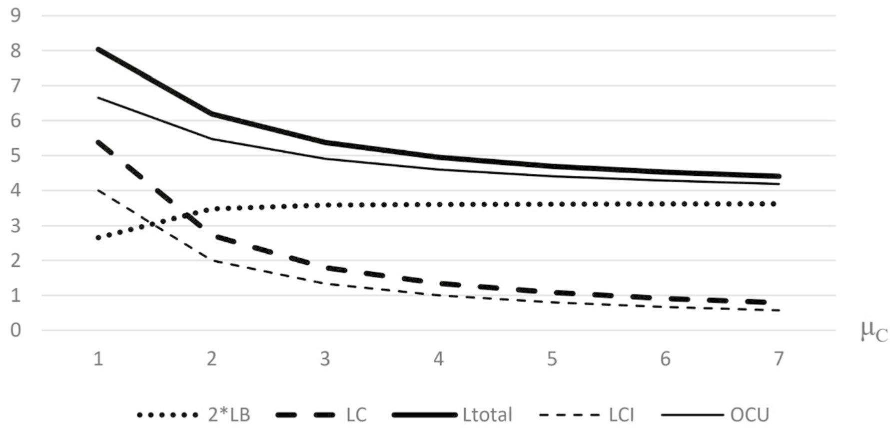

Fig. 5 Number of cars and buses as functions of μ_C when $M = 8, \mu_B = 1, \lambda_B = 2, \lambda_C = 4$

λ_C increases, so do $E[L_C]$, $E[L_{CI}]$, $E[OCI]$ and $E[L_{total}]$, while $2E[L_B]$ decreases. Clearly, when $\lambda_C \to M\mu_C = 8$, $E[OCI]$ approaches $M = 8$.

Figure 4 depicts the same performance measures as in Fig. 3, but as functions of λ_B, when $\lambda_C = 4, \mu_C = 1, \mu_B = 1$. It is seen that $E[L_{CI}] = \frac{\lambda_C}{\mu_C} = 4$ is constant, independent of λ_B, while $2E[L_B]$ approaches 4 when λ_B becomes large.

In Fig. 5, the changing parameter is μ_C. When μ_C increases, all measures decrease, except for $2E[L_B]$ that increases to a maximum $2E[L_B] = 4$.

In Fig. 6, the independent parameter is μ_B. When it becomes large, $2E[L_B]$ approaches zero (buses leave almost instantly after entering). As for the cars, the

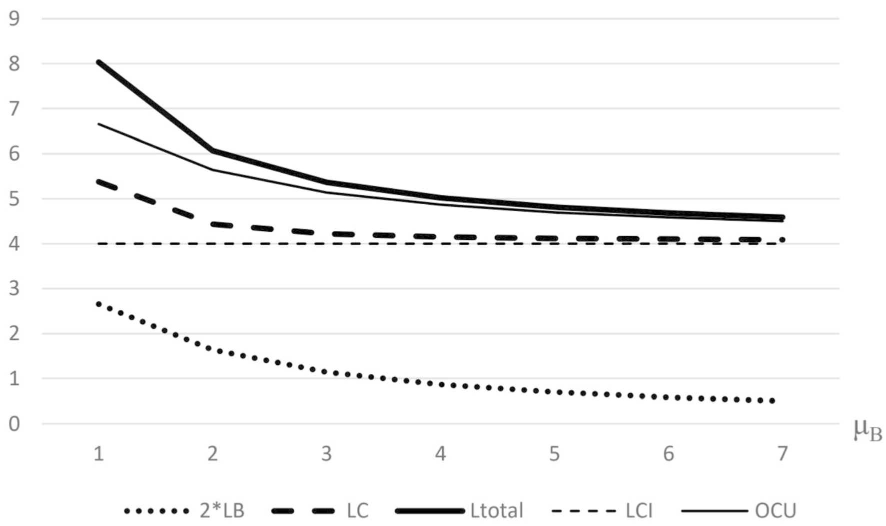

Fig. 6 Number of cars and buses as functions of μ_B when $M = 8, \mu_C = 1, \lambda_B = 2, \lambda_C = 4$

model becomes practically an $M(\lambda_C)/M(\mu_C)/8$ system, so that, when $M\mu_C >> \lambda_C$, $E[L_C^q]$ is small and $E[L_C] \rightarrow E[L_{CI}] = \frac{\lambda_C}{\mu_C} = 4$.

6 Parking lot's optimal size

It is clear that a too large, or a too small, PL size is very costly and non-economical. Consequently, in this section, we determine the optimal size of the PL under conventional economic assumptions. Assume that the utilities of a car and of a bus for using the parking lot are V_C and V_B, respectively. Let r_C and r_B denote the fixed entrance fee to the PL for a car, and for a bus, respectively. It is assumed that $V_C > r_C$ and $V_B > r_B$. Let C_C and C_B, respectively, denote the waiting time cost rate for a car, or for a bus, in the system. Let d_B denote the fixed loss for a bus that is blocked and balks, and let g be the cost per unit time for maintaining a single spot of the PL. The society overall objective is to determine the optimal PL size M^* that maximizes the value of $Z(M)$, where

$$\begin{aligned} Z(M) = \{ &\lambda_C(V_C - r_C) + \lambda_B(1 - P(blocked\ bus))(V_B - r_B) \\ &- E[L_C]C_C - E[L_B]C_B - \lambda_B P(blocked\ bus) d_B - gM \}. \end{aligned} \tag{5}$$

and

$$P(blocked\ bus) = 1 - \sum_{i=0}^{\frac{M}{2}-1} \sum_{j=0}^{M-2i-2} p_{i,j}. \tag{6}$$

Note that $\lambda_C(V_C - r_C)$, as well as $\lambda_B(V_B - r_B)$, do not affect the optimal result of (5). Thus, the objective becomes

$$\min_{M}\{Z_1(M) = \lambda_B P(blocked\ bus))(V_B - r_B + d_B) + E[L_C]C_C + E[L_B]C_B + gM\}. \tag{7}$$

Equation (7) points at the dichotomy: high value of M decreases the values of $P(blocked\ bus)$ and of $E[L_C]$, but increases gM, whereas a small value of M increases both the blocking probability and $E[L_C]$, but decreases gM.

Lemma 6.1 : $\forall g > 0, \forall M > \frac{\lambda_C}{\mu_c}$, *$Z(M)$ has at least one global maximum.*

Proof From (1), if $M > \frac{\lambda_C}{\mu_c}$, the queueing system is stable. Hence, $\forall M > \frac{\lambda_C}{\mu_c}$, $Z(M)$ is finite. In addition, $\lim_{M\to\infty} Z(M) = -\infty$. So, $Z(M)$ has at least one global maximum. □

Figure 7 exhibits the effect of M on $Z(M)$; $Z(M)$ is an increasing function in the range from 6 to 12, gaining its maximal value at $M^* = 12$. Beyond that, $Z(M)$ decreases monotonically.

Equivalence of a fixed charge entering fee and charging by sojourn time: Another common PL operating mode is to charge cars and buses as a function of their actual sojourn time in the PL. Let h_C and h_B, respectively, denote the charging rate (per unit of time) of a car and of a bus. Then, the overall objective is to find M^* so has to

Fig. 7 Total profit as function of M when $\mu_C = 1$, $\mu_B = 1$, $\lambda_B = 1$, $\lambda_C = 4$, $V_C - r_C = 60$, $V_B - r_B = 60$, $C_B = 40$, $C_C = 40$, $g = 5$, $d_B = 5$

$$\max_{M}\{Z_2(M) = [\lambda_C V_C - \lambda_C h_C \frac{1}{\mu_C} + \lambda_B(1 - P(blocked\ bus))V_B$$
$$- \lambda_B(1 - P(blocked\ bus))h_B \frac{1}{\mu_B} - E[L_C]C_C - E[L_B]C_B - \lambda_B P(blocked\ bus)d_B - gM]\}. \tag{8}$$

Examining Eq. (8), it is readily seen that $\frac{h_C}{\mu_C}$ replaces r_C in Eq. (5), and $\frac{h_B}{\mu_B}$ replaces r_B. That is, the two operating modes are practically equivalent depending only on the values r_C versus $\frac{h_C}{\mu_C}$ and r_B versus $\frac{h_B}{\mu_B}$.

7 To split or not?

We now investigate whether or not it is profitable to split the M-spot PL into two independent and separate systems. The 'split or not' question falls into the classical queueing dilemma 'to pool or not' (see e.g., Cao et al. (2020)). Moreover, . Yechiali (1977) compared the relative mean queueing time of a customer in the GI/M/s queue versus the mean queueing time of a customer in a corresponding system of s parallel GI/M/1 queues. In particular, it is shown that the ratio between the latter and the former tends to infinity when the traffic intensity ρ tends to 0, and it approaches s when ρ goes to 1. Thus, it is clear that from pure queueing considerations, a single joint lot is better than two separate ones. However, when costs are involved, the picture may change.

Consider a PL with M parking spots. Suppose that the lot is split into two separate and independent lots with N ($N \leq M$) parking spaces for cars and $\frac{M-N}{2} = K$ (assumed even) double spaces for buses. The corresponding independent arrival processes are Poisson, with rates λ_C and λ_B, respectively. Thus, the car lot is a $M(\lambda_C)/M(\mu_C)/N$ queueing system [with unbounded buffer). It is well known (Cooper 1981) page 97] that the conditional queueing (excluding service) time of a car, W_C^q, given that it has to queue, is exponentially distributed with mean $\frac{1}{N\mu_C - \lambda_C}$. That is,

$$[W_C^q | W_C^q > 0] \sim exp(N\mu_C - \lambda_C).$$

The mean queue size of cars, $E(L_C^q)$, mean queueing time, $E(W_C^q)$, mean total sojourn time, $E(W_C)$, and mean number of cars, $E(L_C)$, in the system are given, respectively, by

$$E[L_C^q] = \frac{a_C^N \cdot \rho_C}{N!(1-\rho_C)^2} \frac{1}{\sum_{n=0}^{N-1} \frac{a_C^n}{n!} + \frac{a_C^N}{N!} \frac{1}{1-\rho_C}},$$

where $a_C = \frac{\lambda_C}{\mu_C}$ and $\rho_C = \frac{a_C}{N}$. Then,

$$E[W_C^q] = \frac{1}{\lambda_C} E(L_C^q),$$

$$E[W_C] = \frac{1}{\lambda_C} E(L_C^q) + \frac{1}{\mu_C},$$

$$E[L_C] = E(L_C^q) + a_C. \tag{9}$$

On the other hand, the bus lot is a $M(\lambda_B)/M(\mu_B)/K/K$, finite space, blocked customers cleared, queueing system. Then, with $a_B = \frac{\lambda_B}{\mu_B}$,

$$E[L_B] = \frac{\sum_{i=1}^{K} \frac{a_B^i}{(i-1)!}}{\sum_{i=0}^{K} \frac{a_B^i}{i!}}, \tag{10}$$

$$E[W_B] = \frac{1}{\lambda_B(1 - P(blocked\ bus))} E[L_B].$$

where $P(blocked\ bus)$ is given by Erlang's loss formula

$$P(blocked\ bus) = \frac{\frac{a_B^K}{K!}}{\sum_{i=0}^{K} \frac{a_B^i}{i!}}. \tag{11}$$

Now, let

$$Z_3(N) = \lambda_C(V_C - r_C) - E[L_C]C_C - gN$$

be the total reward value from the car lot, and

$$Z_4(K) = \lambda_B(1 - P(blocked\ bus))(V_B - r_B) - E[L_B]C_B - \lambda_B P(blocked\ bus)d_B - gK$$

where $P(blocked\ bus)$, $E[L_C]$ and $E[L_B]$ are given, respectively, by Eqs. (11), (9) and (10). Given M, the objective function in this case is

$$min\{Z(M), \min_N\{Z_3(N) + Z_4(M - N)\}\} \tag{12}$$

where $Z(M)$ is given by Eq. (5).

The solid line in Figures 8 and 9 depicts the total profit of the PL owner when the PL is split into N spots for cars and $K = \frac{M-N}{2}$ double spaces for buses, while the dashed line depicts the total profit of the PL owner when the PL is not split. Figure 8 demonstrates a case when it is optimal to split. In this case, the optimal solution is $N = 4$ and $K = 4$. In comparison with this, Fig. 9 demonstrates a case when it is optimal not to split. What makes the difference between the cases is the entrance fee r_B. It follows that if r_B is rather large, it is better not to split the parking lot, but allow more space for buses.

Fig. 8 Total profit of a split parking lot as function of N when $M = 12$, μ_c=1, μ_B= 1, λ_B=2, λ_c =4, $V_C - r_C$=60, $V_B - r_B$= 300, $C_B = 40$, C_C 40, $g = 5$, $d_B = 5$

Fig. 9 Total profit of a split parking lot as function of N when $M = 12$, $\mu_c = 1$, μ_B= 1, λ_B= 2, $\lambda_c = 4$, $V_C - r_C = 60$, $V_B - r_B =$ 600, $C_B = 40$, C_C 40, $g = 5$, $d_B = 5$

8 Conclusion

A model for double-space parking of vehicles (e.g., buses) in a parking lot of finite size parking lot is investigated, where both buses and cars use the same lot. A car occupies a single spot, while a bus occupies double spots. Cars wait in line until entering the PL, but blocked buses leave the system. The probabilistic characteristics of the resulting queueing system are analyzed, and various performance measures are calculated. It is further shown that, from the PL owner point of view, it is equivalent to either charge a fixed entrance fee or charge per unit of time usage. Finally, the optimal split of the PL into two separate and independent parking lots, one for cars only, the other for buses, is investigated and the optimal split is calculated numerically.

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